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Edge-splittings preserving local edge-connectivity of graphs☆

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Abstract

Let $G=(V+s, E)$ be a 2-edge-connected graph with a designated vertex s. A pair of edges rs, st is called admissible if splitting off these edges (replacing rs and st by rt) preserves the local edge-connectivity (the maximum number of pairwise edge disjoint paths) between each pair of vertices in V. The operation splitting off is very useful in graph theory, it is especially powerful in the solution of edge-connectivity augmentation problems as it was shown by Frank [Augmenting graphs to meet edge-connectivity requirements, SIAM J. Discrete Math. 5(1) (1992) 22–53]. Mader [A reduction method for edge-connectivity in graphs, Ann. Discrete Math. 3 (1978) 145–164] proved that if $d(s) \neq 3$ then there exists an admissible pair incident to s. We generalize this result by showing that if $d(s) \geqslant 4$ then there exists an edge incident to s that belongs to at least $\lfloor d(s)/3 \rfloor$ admissible pairs. An infinite family of graphs shows that this bound is best possible. We also refine a result of Frank [On a theorem of Mader, Discrete Math. 101 (1992) 49–57] by describing the structure of the graph if an edge incident to s belongs to no admissible pairs. This provides a new proof for Mader's theorem.

Keywords: Local edge-connectivity; Splitting off; Edge-connectivity augmentation

1. Introduction

In this paper, $G=(V+s, E)$ denotes a 2-edge-connected graph, s being a vertex not in V. (It would be enough to suppose that no cut edge is incident to s but for the sake of simplicity we suppose that G contains no cut edge at all.)

For two vertices $u, v \in V$, the *local edge-connectivity*, $\lambda_G(u, v)$, between u and v is the maximum number of edge disjoint paths between u and v. If $\lambda_G(u, v) \geqslant k$ for all pairs $u, v \in V$, then G is called *k-edge-connected in V*.

The operation *splitting off* is defined as follows: two edges rs and st are replaced by a new edge rt. The graph obtained from G by splitting off a pair of edges rs, st is denoted by G_{rt}. A pair of edges rs, st is called *k-admissible* if G_{rt} is k-edge-connected in V. The pair of edges rs, st is called *admissible* if $\lambda_{G_{rt}}(u, v) \geqslant \lambda_G(u, v)$ for all pairs $u, v \in V$. An edge incident to s is called *admissible* if it belongs to an admissible pair, otherwise it is called *non-admissible*.

The first splitting off result is due to Lovász [6].

Theorem 1.1. *If $G=(V+s, E)$ is k-edge-connected in V for some $k \geqslant 2$ and $d(s)$ is even then each edge incident to s belongs to a k-admissible pair.*

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Cai and Sun [3] showed how to apply this result to solve the following global edge-connectivity augmentation problem: given a graph H and an edge-connectivity requirement $k \in \mathbb{Z}_+$, find the minimum number of new edges whose addition makes the graph k-edge-connected.

Theorem 1.1 was extended in Bang-Jensen et al. [1].

Theorem 1.2. *If $G=(V+s,E)$ is k-edge-connected in V for some $k \geqslant 2$ and $d(s)$ is even then each edge incident to s belongs to at least $d(s)/2$ (resp. $d(s)/2-1$) k-admissible pairs if k is even (resp. odd).*

In [1], we applied Theorem 1.2 to solve the global edge-connectivity augmentation problem in bipartite graphs: given a connected bipartite graph H and an edge-connectivity requirement $k \in \mathbb{Z}_+$, what is the minimum number of new edges whose addition results in a bipartite k-edge-connected graph.

It is easy to construct examples to show that the bounds of Theorem 1.2 are best possible.

Mader [7] generalized Theorem 1.1 on local edge-connectivity.

Theorem 1.3. *If $G=(V+s,E)$ is 2-edge-connected and $d(s) \neq 3$ then there exists an admissible pair incident to s.*

Applying this result, Frank [5] solved the local edge-connectivity augmentation problem: given a graph $H=(V,E)$ and a requirement function $r\colon V \times V \to \mathbb{Z}_+$, find the minimum number of new edges F such that $\lambda_{H+F}(u,v) \geqslant r(u,v)$ for all pairs $u, v \in V$.

The main contribution of the present paper is the following strengthening of Theorem 1.3. It can be considered as the counterpart of Theorem 1.2 for local edge-connectivity.

Theorem 1.4. *If $G=(V+s,E)$ is a 2-edge-connected graph and $d(s) \geqslant 4$ then there is an edge sr that belongs to at least $\lfloor d(s)/3 \rfloor$ admissible pairs incident to s.*

We present, in Section 3, an infinite family of graphs showing that our bound is best possible.

Theorem 1.3 implies that at most three edges incident to s are non-admissible. Frank [5] provided a slight generalization of this result.

Theorem 1.5. *If $G=(V+s,E)$ is 2-edge-connected and $d(s) \neq 3$ then at most one edge incident to s belongs to no admissible pair.*

We refine this result by describing the structure of the graph if it contains a non-admissible edge incident to s. (For definitions, see Section 2.)

Theorem 1.6. *Let st be an edge of a 2-edge-connected graph $G=(V+s,E)$. The following are equivalent:*

(a) *The edge st is non-admissible.*
(b) *There exist two dangerous sets M_1 and M_2 such that $t \in M_1 \cap M_2$ and $M_1 \cup M_2$ contains all the neighbours of s.*
(c) *The degree $d(s)$ of s is odd and there exist two disjoint tight sets C_1 and C_2 in $V-t$ such that $d(s,C_1)=d(s,C_2)=(d(s)-1)/2$.*

As an application of Theorem 1.6 we present the following result.

Theorem 1.7. *Let $G=(V+s,E)$ be a 2-edge-connected graph with $d(s) \neq 3$. If an edge st is non-admissible then each edge $sr \neq st$ belongs to exactly $(d(s)-1)/2$ admissible pairs.*

The proofs of Theorems 1.6 and 1.7, given in Sections 4 and 5, together provide a new proof of Theorem 1.5 and hence of Theorem 1.3.

We mention a related interesting result of Bang-Jensen and Jordán [2].

Theorem 1.8. *Let $G=(V+s,E)$ be a 2-edge-connected graph. Then, for every edge st, the number of edges rs for which the pair of edges rs, st is non-admissible is at most $2k^2-2k$, where $k=\max\{\lambda_G(u,v) : u, v \in V\}$.*

2. Notation and preliminary results

Let $G=(V+s,E)$ be a graph, with s a vertex not in V. Let $\Gamma(s)$ denote the set of neighbours of s. We use the notation $\subset$ for proper subset. For a set $T\subset V$, $T\neq\emptyset$, we denote the graph obtained from G by contracting T into one vertex v_T by G/T.

Let $X,Y\subseteq V+s$. Let $d(X,Y)$ denote the number of edges between $X-Y$ and $Y-X$. Let $\overline{d}(X,Y)$ denote the number of edges between $X\cap Y$ and $V+s-(X\cup Y)$. We define the degree of the set X by $d(X)=d(X,V+s-X)$. The degree function satisfies the following two well-known equalities:

(1) $d(X)+d(Y)=d(X\cap Y)+d(X\cup Y)+2d(X,Y)$,
(2) $d(X)+d(Y)=d(X-Y)+d(Y-X)+2\overline{d}(X,Y)$.

Observe that, by Menger's theorem, $\lambda_G(x,y)=\lambda(x,y)=\min\{d(Z): Z\subset V+s, x\in Z, y\notin Z\}$ for all $x,y\in V$. We define the function $R(X)$ as follows: $R(\emptyset)=R(V)=0$ and, for a set $X\subset V$, $X\neq\emptyset$, let

$$R(X)=\max\{\lambda_G(x,y): x\in X, y\in V-X\}.$$

Observe that the function $R(X)$ satisfies (3) and (4) for $X,Y\subset V$:

(3) $R(X)=R(V-X)$,
(4) $R((X-Y)\cup(Y-X))\leqslant\max\{R(X-Y),R(Y-X)\}$.

The following property of $R(X)$ can be found in [4, Proposition 5.4]: for $X,Y\subset V$, at least one of (5) and (6) hold. If $X\cup Y=V$ then (2) holds:

(5) $R(X)+R(Y)\leqslant R(X\cap Y)+R(X\cup Y)$,
(6) $R(X)+R(Y)\leqslant R(X-Y)+R(Y-X)$.

Finally, we define the function

$$h(X):=d(X)-R(X).$$

Note that the function $h(X)$ satisfies (7) and (8) for $X,Y\subset V$.

(7) $h(X)\geqslant 0$,
(8) $h(X)=h(V-X)+2d(s,X)-d(s)$.

The properties above imply:

Proposition 2.1. *For $X,Y\subset V$, at least one of* (9) *and* (10) *hold. If $X\cup Y=V$ then* (10) *holds*:

(9) $h(X)+h(Y)\geqslant h(X\cap Y)+h(X\cup Y)+2d(X,Y)$,
(10) $h(X)+h(Y)\geqslant h(X-Y)+h(Y-X)+2\overline{d}(X,Y)$.

A set $\emptyset\neq X\subset V$ is called *tight* if $h(X)=0$ and it is called *dangerous* if $h(X)\leqslant 1$. Note that tight and dangerous sets are, by definition, subsets of V.

The following claim is due to Mader.

Claim 2.2. *Let T be a tight set in a graph $G=(V+s,E)$ and $G':=G/T$*:

(a) *If a pair of edges e', f' incident to s is admissible in G' then the corresponding pair of edges e, f is admissible in G.*
(b) *If $X'\subseteq V(G')-s$ then $h_{G'}(X')=h_G(X)$, where $X=X'-v_T\cup T$ if $v_T\in X'$ and $X=X'$ otherwise.*

The reduction method of Claim 2.2 will be applied in our proofs and hence we will be able to assume that:

(11) every tight set is a singleton.

We need the following claims.

Claim 2.3 (*Frank* [*5, Claim 3.1*]). *A pair of edges us, sv of a graph $G = (V + s, E)$ is admissible if and only if there is no dangerous set M with $u, v \in M$.*

Claim 2.4 (*Frank* [*5, Claim 4.1*]). *Let $G = (V + s, E)$ be a graph and $t \in \Gamma(s)$ be a vertex of minimum degree. Suppose that* (11) *holds. If a set $M \subseteq V$ contains t and $|\Gamma(s) \cap M| \geqslant 2$, then $R(M - t) \geqslant R(M)$.*

Claim 2.5 (*Bang-Jensen and Jordán* [*2, Lemma 5.4*]). *Let $G = (V + s, E)$ be a 2-edge-connected graph. If M is a dangerous set then*

(a) *$d(s, M) \leqslant (d(s) + 1)/2$, with equality only if $V - M$ is tight, and*
(b) *$d(X, M - X) \geqslant 1$ for every $\emptyset \neq X \subset M$.*

Proof. (a) By (8), since M is dangerous and by applying (8) for $V - M$, $d(s, M) = (d(s) + h(M) - h(V - M))/2 \leqslant (d(s) + 1)/2$ and (a) follows. □

We close this section with a technical lemma.

Lemma 2.6. *Let $G = (V + s, E)$ be a 2-edge-connected graph, $st \in E$ and $S \subseteq V$. Let $\mathscr{M}$ be a minimum collection of dangerous sets such that $t \in \cap\mathscr{M}$ and $S \subseteq \cup\mathscr{M}$. If $|\mathscr{M}| \geqslant 3$,* (11) *holds and $M_i, M_j \in \mathscr{M}$, then*:

(a) (*Bang-Jensen and Jordán* [*2, Lemma 5.4*]) *Property* (10) *does not apply for M_i and M_j, and*
(b) *$M_i \cap M_j = t$.*

Proof. (a) Suppose that (10) applies for M_i and M_j. Then, by $1 \geqslant h(M_i)$ and $1 \geqslant h(M_j)$, we have $h(M_i - M_j) = 0$ and $h(M_j - M_i) = 0$ (so by (11), $M_i - M_j = r_i$ and $M_j - M_i = r_j$ for some vertices $r_i, r_j \in V$) and $\overline{d}(M_i, M_j) = 1$. Let $M_k \in \mathscr{M} - \{M_i, M_j\}$ and $X = M_i \cap M_j \cap M_k$. Note that $t \in X$ so $X \neq \emptyset$. By the minimality of $\mathscr{M}$, $M_k - X \neq \emptyset$. Then, by Claim 2.5(b) and since st enters $M_i \cap M_j$, we have $1 \leqslant d(X, M_k - X) \leqslant d(M_i \cap M_j, M_k - (M_i \cap M_j)) \leqslant \overline{d}(M_i, M_j) - d(M_i \cap M_j, s) \leqslant 1 - 1 = 0$, a contradiction.

(b) By Proposition 2.1 and (a), (9) applies for M_i and M_j. Then, since $1 \geqslant h(M_i)$, $1 \geqslant h(M_j)$ and by the minimality of $\mathscr{M}$, $h(M_i \cup M_j) \geqslant 2$ (otherwise we could replace M_i and M_j by $M_i \cup M_j$), we have $h(M_i \cap M_j) = 0$ and, hence, by (11) and $t \in M_i \cap M_j$, (b) is satisfied. □

3. Proof of Theorem 1.4

The proof is similar to the proof of Theorem 1.3 given by Frank in [5].

Proof. We prove the theorem by induction on $|V|$. We may assume, by Claim 2.2(a), that (11) is satisfied. Let t be a neighbour of s of minimum degree. Let S be the set of neighbours r of s such that $r = t$ or the pair of edges rs, st is not admissible. By Claim 2.3, there is a minimum collection $\mathscr{M}$ of dangerous sets such that $t \in \cap\mathscr{M}$ and $S \subseteq \cup\mathscr{M}$. Suppose that st belongs to less than $\lfloor d(s)/3 \rfloor$ admissible pairs (otherwise, we are done). Then:

(12) $d(s, \cup\mathscr{M}) \geqslant d(s, S) > d(s) - \lfloor d(s)/3 \rfloor = \lceil 2d(s)/3 \rceil$.

By Claim 2.5(a) and (12), for $M_i \in \mathscr{M}$, $d(s, M_i) \leqslant (d(s) + 1)/2 < \lceil 2d(s)/3 \rceil < d(s, \cup\mathscr{M})$ and hence $|\mathscr{M}| \geqslant 2$. Let $M_1, M_2 \in \mathscr{M}$. By the minimality of $\mathscr{M}$, each $M_i \in \mathscr{M}$ contains a neighbour $r_i \neq t$ of s that belongs to no other $M_j \in \mathscr{M}$. Let us choose such a vertex r_i for each $M_i \in \mathscr{M}$.

Claim 3.1. $\mathcal{M} = \{M_1, M_2\}$.

Proof. For $i = 1, 2$, M_i contains t and r_i, so $|\Gamma(s) \cap M_i| \geqslant 2$. Then, by Claim 2.4, $R(M_1 - t) \geqslant R(M_1)$ and $R(M_2 - t) \geqslant R(M_2)$. Suppose that $|\mathcal{M}| \geqslant 3$. Then, by Lemma 2.6(b), $M_1 \cap M_2 = t$, thus M_1 and M_2 satisfy (6) and hence (10), a contradiction by Lemma 2.6(a). □

Claim 3.2. *Property* (10) *applies for* M_1 *and* M_2.

Proof. Suppose that (10) does not hold for M_1 and M_2. Then, by Proposition 2.1, $M_1 \cup M_2 \neq V$ and (9) applies for M_1 and M_2. By (8), (7), Claim 3.1, (12) and $d(s) \geqslant 4$, $h(M_1 \cup M_2) \geqslant 2d(s, M_1 \cup M_2) - d(s) = 2d(s, \cup\mathcal{M}) - d(s) > 2\lceil 2d(s)/3 \rceil - d(s) \geqslant 2$.

It follows, by $1 \geqslant h(M_1)$, $1 \geqslant h(M_2)$, (9) and (7), that $1+1 \geqslant h(M_1) + h(M_2) \geqslant h(M_1 \cap M_2) + h(M_1 \cup M_2) > 0 + 2$, a contradiction. □

Claim 3.3. $d(s, r_1) + d(s, r_2) \geqslant \lceil 2d(s)/3 \rceil$.

Proof. By $1 \geqslant h(M_1)$, $1 \geqslant h(M_2)$, Claim 3.2, (7), $st \in E$ and $t \in M_1 \cap M_2$, we have $1+1 \geqslant h(M_1) + h(M_2) \geqslant h(M_1 - M_2) + h(M_2 - M_1) + 2\overline{d}(M_1, M_2) \geqslant 0 + 0 + 2d(s, M_1 \cap M_2) \geqslant 2$, so $h(M_1 - M_2) = 0 = h(M_2 - M_1)$ and $d(s, M_1 \cap M_2) = 1$. It follows, by $r_1 \in M_1 - M_2$, $r_2 \in M_2 - M_1$ and (11), that $M_1 - M_2 = r_1$ and $M_2 - M_1 = r_2$. Then, by Claim 3.1 and (12), $d(s, r_1) + d(s, r_2) = d(s, M_1 \cup M_2) - d(s, M_1 \cap M_2) = d(s, \cup\mathcal{M}) - 1 \geqslant \lceil 2d(s)/3 \rceil$. □

Let e_i be any edge connecting s and r_i for $1 \leqslant i \leqslant 2$.

Claim 3.4. *The pair of edges* e_1, e_2 *is admissible*.

Proof. Otherwise, by Claim 2.3, there is a dangerous set X with $r_1, r_2 \in X$, and then, by (8), (7), Claim 3.3 and $d(s) \geqslant 4$, we have $1 \geqslant h(X) \geqslant 2d(s, X) - d(s) \geqslant 2\lceil 2d(s)/3 \rceil - d(s) \geqslant 2$, a contradiction.

By Claim 3.3, we may assume without loss of generality that $d(s, r_1) \geqslant \lceil d(s)/3 \rceil \geqslant \lfloor d(s)/3 \rfloor$. Then, by Claim 3.4, e_2 belongs to at least $\lfloor d(s)/3 \rfloor$ admissible pairs and the proof of Theorem 1.4 is complete. □

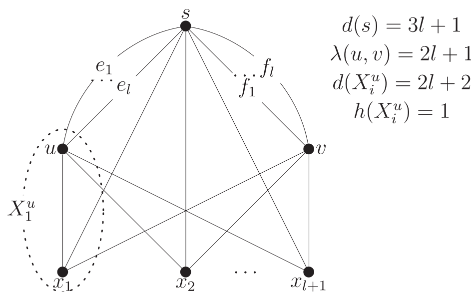

Fig. 1. Each edge incident to s belongs to exactly $\lfloor d(s)/3 \rfloor$ admissible pairs.

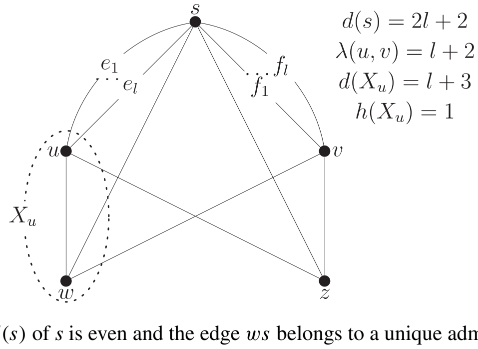

Fig. 2. The degree $d(s)$ of s is even and the edge ws belongs to a unique admissible pair ws, sz.

Examples. There exists an infinite class of graphs in which each edge incident to s belongs to exactly $\lfloor d(s)/3 \rfloor$ admissible pairs. See Fig. 1. We mention that it is not true in general, even if we suppose that the degree of s is even, that each edge incident to s belongs to many admissible pairs. In Fig. 2, the edge ws belongs to the unique admissible pair of edges ws, sz.

4. Proof of Theorem 1.6

Proof. We consider first the most complicated part, we prove that (a) implies (b) by induction on $|V|$.

Claim 4.1. *We may assume that* (11) *is satisfied.*

Proof. Suppose that there exists a tight set T with $|T| > 1$. Let $G' = G/T$. By Claim 2.2(a), st belongs to no admissible pair in G', G' is 2-edge-connected and $|V(G')| < |V|$; hence, by induction, (b) is true for G' and, then, by Claim 2.2 (b), it is also true for G. $\square$

The edge st belongs to no admissible pair; thus, by Claim 2.3, there is a minimum collection $\mathscr{M}$ of dangerous sets such that $t \in \cap\mathscr{M}$ and $\Gamma(s) \subseteq \cup\mathscr{M}$. By the minimality of $\mathscr{M}$, each $M_i \in \mathscr{M}$ contains a neighbour $r_i \neq t$ of s that belongs to no other $M_j \in \mathscr{M}$. Let us choose such a vertex r_i for each $M_i \in \mathscr{M}$. By Claim 2.5(a), $d(s) \geqslant 2$ and $\Gamma(s) \subseteq \cup\mathscr{M}$, for $M_i \in \mathscr{M}$, $d(s, M_i) \leqslant (d(s)+1)/2 < d(s) = d(s, \cup\mathscr{M})$ and hence $|\mathscr{M}| \geqslant 2$.

Suppose that $|\mathscr{M}| \geqslant 3$. We shall find a contradiction showing that this case can not happen and hence $|\mathscr{M}| = 2$. By Lemma 2.6(b), for all $M_i, M_j \in \mathscr{M}$, $M_i - M_j = M_i - t$. Let $T = V - \cup\mathscr{M}$. Note that $d(s, T) = 0$.

Claim 4.2. *If $R(M_1) = \lambda(a, b)$ with $a \in M_1$ and $b \in T$, then for some $M_k \in \mathscr{M} - M_1$, $R(M_k - t) > R(t)$.*

Proof. Note that $d(s) \geqslant |\mathscr{M}| + 1$ and $d(T) \geqslant \lambda(a, b) = R(M_1) \geqslant d(M_1) - 1$ because M_1 is dangerous. By repeated applications of (1) we get

$$\begin{aligned}\sum_{M_j \in \mathscr{M}} (d(M_j) - d(t)) &\geqslant d(s \cup T) - d(t)\\ &= d(s) + d(T) - d(t)\\ &\geqslant (|\mathscr{M}| + 1) + (d(M_1) - 1) - d(t)\\ &> (|\mathscr{M}| - 1) + (d(M_1) - d(t)),\end{aligned}$$

so there exists $M_k \in \mathscr{M} - M_1$ with $d(M_k) - d(t) > 1$. Then, since M_k is dangerous, $R(M_k) \geqslant d(M_k) - 1 > d(t) \geqslant R(t)$, so, by (4), $R(M_k - t) > R(t)$. $\square$

Claim 4.3. *There exists $M_i \in \mathscr{M}$ for which $R(M_i - t) \geqslant R(t)$.*

Proof. Let $Y = \{y \in V - t : R(t) = \lambda(t, y)\}$. By definition, $Y \neq \emptyset$. If there exists a vertex $y \in M_i \cap Y$ for some $M_i \in \mathscr{M}$, then $R(M_i - t) \geqslant \lambda(t, y) = R(t)$. Thus we may suppose that $Y \subseteq T$. Let $y \in Y$. Then $R(M_1) \geqslant \lambda(t, y) = R(t)$. If $R(M_1) = \lambda(t, y)$ then, by Claim 4.2, $R(M_1 - t) > R(t)$. Otherwise $R(M_1) > R(t)$, so, by (4), $R(M_1 - t) > R(t)$. $\square$

Claim 4.4. *If $M_j \in \mathscr{M} - M_i$, then $R(M_j - t) < R(M_j) \leqslant R(t)$.*

Proof. Suppose that $R(M_j - t) \geqslant R(M_j)$. By Claim 4.3 and (4), $R(M_i - t) \geqslant R(M_i)$. So (6) and hence (10) apply for M_i and M_j, contradicting Lemma 2.6(a). By $R(M_j - t) < R(M_j)$ and (4), $R(M_j) \leqslant R(t)$. $\square$

Claim 4.5. *If $R(M_i) = \lambda(a, b)$ with $a \in M_i$ and $b \in V - M_i$, then $b \in T$.*

Proof. Suppose that $b \in M_j \in \mathscr{M} - M_i$. Then, $R(M_j - t) \geqslant \lambda(a, b) = R(M_i)$. By Claims 4.4 and 4.3, $R(M_j) \leqslant R(t) \leqslant R(M_i - t)$. Thus (6) and hence (10) apply for M_i and M_j, a contradiction by Lemma 2.6(a). $\square$

By Claims 4.3 and 4.4, there exists $M_i \in \mathscr{M}$ such that $R(M_j - t) < R(t)$ for all $M_j \in \mathscr{M} - M_i$. However, by Claims 4.5 and 4.2, applied for $M_1 = M_i$, $R(M_j - t) > R(t)$ for some $M_j \in \mathscr{M} - M_i$. This contradiction completes the proof of (a) implies (b).

Obviously, (b) implies (a) by Claim 2.3.

We show now that (b) implies (c). Let $C_1 = M_1 - M_2$ and $C_2 = M_2 - M_1$. Clearly, $C_1 \cap C_2 = \emptyset$ and, by $t \in M_1 \cap M_2$, the sets C_1 and C_2 are in $V - t$.

Claim 4.6. *$d(s)$ is odd and $d(s, C_1) = (d(s) - 1)/2 = d(s, C_2)$.*

Proof. By (8), $\Gamma(s) \subseteq M_1 \cup M_2$ and $st \in E$, we have $2(d(s) + 1)/2 \geqslant d(s, M_1) + d(s, M_2) = d(s, M_1 \cup M_2) + d(s, M_1 \cap M_2) \geqslant d(s) + 1$. It follows that $d(s)$ is odd, $d(s, M_i) = (d(s) + 1)/2$ and $d(s, M_1 \cap M_2) = 1$. Then $d(s, C_i) = d(s, M_i) - d(s, M_1 \cap M_2) = (d(s) + 1)/2 - 1 = (d(s) - 1)/2$ for $i = 1, 2$. □

Claim 4.7. *Property* (10) *applies for M_1 and M_2.*

Proof. Suppose that (10) does not hold for M_1 and M_2. Then, by Proposition 2.1, $M_1 \cup M_2 \neq V$ and (9) applies for M_1 and M_2, so, by $1 \geqslant h(M_1)$, $1 \geqslant h(M_2)$ and (7), we have $2 \geqslant h(M_1 \cup M_2)$. It follows, by (8), (7) and $\Gamma(s) \subseteq M_1 \cup M_2$, that $2 \geqslant h(M_1 \cup M_2) = h(V - (M_1 \cup M_2)) + 2d(s, M_1 \cup M_2) - d(s) \geqslant d(s)$. However, since G is 2-edge-connected and $d(s)$ is odd, $d(s) \geqslant 3$, a contradiction. □

Then, by $1 \geqslant h(M_1)$, $1 \geqslant h(M_2)$, (10), $t \in M_1 \cap M_2$ and $st \in E$, we get that $h(C_1) = 0 = h(C_2)$, that is, C_1 and C_2 are tight sets. This completes the proof of (b) implies (c).

Finally, we show that (c) implies (b). Suppose that $d(s)$ is odd and there exist two disjoint tight sets $C_1, C_2 \subseteq V - t$ such that $d(s, C_1) = (d(s) - 1)/2 = d(s, C_2)$. Then, by (8), $M_1 = V - C_1$ and $M_2 = V - C_2$ are dangerous sets. Note that $t \in M_1 \cap M_2$ and $\Gamma(s) \subseteq M_1 \cup M_2$. □

5. Proof of Theorem 1.7

By Theorem 1.6, there exist two dangerous sets M_1 and M_2 with $t \in M_1 \cap M_2$ and $\Gamma(s) \subseteq M_1 \cup M_2$. It also follows from the proof above that $d(s, M_1 \cap M_2) = 1$ and $d(s, M_1) = d(s, M_2) = (d(s) + 1)/2$. Let $sr \neq st$ be an edge incident to s. Then, by Claim 2.3, the edge sr belongs to at most $d(s) - (d(s) + 1)/2 = (d(s) - 1)/2$ admissible pairs. To finish the proof we show the following lemma.

Lemma 5.1. *The edge sr belongs to at least $(d(s) - 1)/2$ admissible pairs.*

Proof. We prove the lemma by induction on $|V|$. We may assume, by Claim 2.2(a), that (11) is satisfied. By Theorem 1.6, $d(s)$ is odd and there exist two disjoint tight sets $C_1, C_2 \subseteq V - t$ such that $d(s, C_1) = d(s, C_2) = (d(s) - 1)/2$. Then, by (11), $C_1 = c_1$ and $C_2 = c_2$ for some vertices $c_1, c_2 \in V$. Since $sr \neq st$, either $r = c_1$ or c_2. The lemma follows from the following claim.

Claim 5.2. *Let e_i be any edge connecting s and c_i for $1 \leqslant i \leqslant 2$. Then the pair of edges e_1, e_2 is admissible.*

Proof. Otherwise, by Claim 2.3, there is a dangerous set X containing c_1 and c_2. Then, by $d(s, c_1) = d(s, c_2) = (d(s) - 1)/2$ and Claim 2.5(a), $2(d(s) - 1)/2 \leqslant d(s, X) \leqslant (d(s) + 1)/2$, that is, $d(s) \leqslant 3$. However, since G is 2-edge-connected and $d(s)$ is odd and $\neq 3$, $d(s) \geqslant 5$, a contradiction. □

6. Open problems

For a summary on edge-connectivity augmentation problems in graphs we refer to [8]. We repeat one of the open problems proposed in [8], the problem of *local edge-connectivity augmentation in bipartite graphs*: given a connected bipartite graph $H = (V, E)$ and a requirement function $r: V \times V \to \mathbb{Z}_+$, find the minimum number of new edges F

such that $\lambda_{H+F}(u, v) \geqslant r(u, v)$ for all pairs $u, v \in V$ and $H + F$ is a bipartite graph. Theorem 1.4 could help to solve this problem.

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